VARIABLE DIMENSION COMPLEXES, PART II:

A UNIFIED APPROACH TO SOME COMBINATORIAL

LEMMAS IN TOPOLOGY

by

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Abstract

Part II of this study uses the path-following theory of labelled V-complexes developed in Part I to provide constructive algorithmic proofs of a variety of combinatorial lemmas in topology. We demonstrate two new dual lemmas on the n-dimensional cube, and use a Generalized Sperner Lemma to prove a generalization of the Knaster-Kuratowski-Mazurkiewicz Covering Lemma on the simplex. We also show that Tucker's Lemma can be derived directly from the Borsuk-Ulam Theorem. We report the interrelationships between these results, Brouwer's Fixed Point Theorem, and the existence of stationary points on the simplex.

Introduction

Part II of this study uses the path-following theory of labelled V-complexes developed in Part I to provide constructive algorithmic proofs of a variety of combinatorial lemmas in topology. We demonstrate two new dual lemmas on the n-dimensional cube, and use a Generalized Sperner Lemma to prove a generalization of the Knaster-Kuratowski-Mazurkiewicz Covering Lemma on the simplex. We also show that Tucker's Lemma can be derived directly from the Borsak-Ulam Theorem. We report the interrelationships between these results, the Brouwer Fixed Point Theorem, the Borsak-Ulam Theorem, the existence of stationary points on the simplex, and two new theorems on the simplex relating to the Generalized Covering Theorem.

In Section I, we give constructive proofs of the Sperner Lemma, Scarf's Dual Sperner Lemma, and a Generalized Sperner Lemma. We show that the Generalized Sperner Lemma leads to simple proofs of the Sperner Lemma and the Dual Sperner Lemma. We show how each of these results is related to Brouwer's Fixed Point Theorem.

Section II introduces a Generalized Covering Theorem that generalizes the Knaster-Kuratowski-Mazurkiewicz Lemma [12]. The Generalized Covering Lemma is used to provide a proof of the existence of stationary points on the simplex, as well as two new results on the simplex.

Section III deals with combinatorial lemmas on the cube that relate to Brouwer's Theorem. Two new dual lemmas on the cube are introduced and given constructive proofs. We also give constructive proofs of Gale's Hex Theorem and Kuhn's Strong Cubical Lemma. It is shown that the Hex Theorem and one of the new lemmas are equivalent. Finally, the interrelationship between these results and Brouwer's Theorem are exposited.

In Section IV, we give a constructive proof of Tucker's Combinatorial Lemma on the cube, and show that the Borsak-Ulam Theorem leads to a direct proof of this lemma.

Notation

In addition the notation used in Part I, let $S^{n-1} = \{x \in IR^n_+ | e^t x = I\}$, the standard $(n-1)$-simplex. Let $C^n = \{x \in IR^n \mid 0 \leq x \leq e\}$, the unit n-cube. Let $\langle v^1, \ldots, v^m \rangle$ denote the convex hull of $v^1, \ldots, v^m$.

Let H be a set in IR^n and let C be a locally finite triangulation of H. By the pseudomanifold corresponding to C we mean the pseudomanifold $K = \{x = \{v^1, \ldots, v^m\} | \langle v^1, \ldots, v^m \rangle$ is a face of some simplex σ of C} and $K^0 = \{v | v$ is a vertex of some simplex σ of C}, where if H is 0-dimensional, then K includes $\emptyset$, the empty set, as an element.

I.   Combinatorial Lemmas on the Simplex

In this section, we give constructive proofs of three combinatorial lemmas on the simplex, Sperner's Lemma [17], Scarf's Dual Sperner Lemma [15], and a Generalized Sperner Lemma. The Generalized Sperner Lemma was independently developed by Ky Fan [3] and the author [4], and inadvertantly by Luthi [14]. We show the interrelationships between these lemmas and the Brouwer Fixed Point Theorem [1].

Our method of proof will be consistent throughout. Given a simplex S^{n-1} and a triangulation C of S^{n-1}, we will define a V-complex, where each pseudomanifold A(T) corresponds to a face of S^{n-1}. Given a labelling $L(\cdot)$ of the vertices of C, we will examine the sets B and G associated with the V-complex. The elements of $B \cup G \setminus \{\emptyset\}$ will be shown to have the desired

properties of the conclusions of the lemmas. By lemma 13 of Part I, $B \cup G \setminus \emptyset$ must have an odd number of elements.

Let C be a locally finite triangulation of S^{n-1} and let K be the complex (an (n-1)-pseudomanifold) corresponding to C, where K^0 is the collection of vertices o f C.

Let $L(\cdot): K^0 \to \{1, \ldots, n\}$ be a labelling of K^0. For $x \in K$, x is called <u>completely labelled</u> if $L(x) = \{1, \ldots, n\}$. The <u>carrier</u> of x, denoted F(x), is the set $\{i | v_i > 0$ for some $v \in x\}$. F(x) is the "index set" of the smallest face of S^{n-1} containing x. If $L(x) = F(x)$, x is called <u>completely labelled in its face</u>.

<u>Sperner's Lemma [17]</u>. Let $L(\cdot): K^0 \to \{1, \ldots, n\}$ be a labelling of K^0 such that if $v \in K^0$ and $v_i = 0$, then $L(v) \neq i$ (such a labelling is called a <u>proper</u> labelling). Then there exists an odd number of completely labelled simplices $x \in K$.

PROOF: Let $N = \{1, \ldots, n\}$, and $\mathcal{J} = \{\{\emptyset\}, \{1\}, \{1,2\}, \{1,2,3\}, \ldots, \{1,2,3, \ldots, n-1\}\}$. Define $A(\emptyset) = \{\emptyset, e^1\}$. For $\emptyset \neq T \in \mathcal{J}$, $T = \{1, \ldots, m\}$ for some $m \leq n-1$, and define A(T) to be the pseudomanifold corresponding to the restriction of C to $\{v \in S^{n-1} | v_i = 0$ for $i > m+1\}$. $K, \mathcal{J}, A(\cdot)$, and N constitute a V-complex. Let $L(\cdot)$ be a proper labelling.

We first examine B. $\emptyset \in B$, since $A(\emptyset) = \{\emptyset, \{e^1\}\}$ contains only one 0-simplex. Suppose $\emptyset \neq x \in B$. Then $L(x) = T_x$ and $x \in \partial' A(T_x)$. Let $T_x = \{1, \ldots, m\}$. $x \in \partial' A(T_x)$ implies that there is an i, $1 \leq i \leq m$, such that $v_i = 0$ for all $v \in x$. Thus $i \notin L(x)$, since $L(\cdot)$ is proper. But then $L(x) \neq T_x$, contradicting the definition of B. Therefore $x \notin B$, whence $B = \{\emptyset\}$.

Now let us examine G. If $x \in G$, $L(x) \supset T_x$, $L(x) \notin \mathcal{J}$. Again let $T_x = \{1, \ldots, m\}$. Because $L(\cdot)$ is proper, $L(x) \subset \{1, \ldots, m+1\}$. Since $T_x \in \mathcal{J}$ and $L(x) \notin \mathcal{J}$, $L(x) = \{1, \ldots, n\}$. Thus x is completely labelled.

Because $|B| = 1$, $|G|$ must be odd. Therefore there are an odd number of completely labelled simplices. $\otimes$

Kuhn's algorithm for Sperner's Lemma [11], independently developed by Shapley [16], consists of tracing the path of adjacent simplices from $\emptyset \in B$ to an element $x \in G$.

Sperner's Lemma requires that the labelling $L(\cdot)$ be restricted on the boundary of S^{n-1}. If we omit this restriction, we obtain the following generalization of Sperner's Lemma:

Generalized Sperner Lemma [3, 4, 14]. Let $L(\cdot): K^0 \to \{1, \ldots, n\}$. Then there exists an odd number of simplices $x \in K$ such that x is completely labelled in its face.

PROOF: As in the proof of Sperner's Lemma, we first construct a V-complex. Let $N = \{1, \ldots, n\}$ and let $\mathcal{J} = \{T \subset N \mid n \notin T\}$. Define $A(\emptyset) = \{\emptyset, \{e^n\}\}$, and for $\emptyset \neq T \in \mathcal{J}$, define $A(T)$ to be the pseudomanifold corresponding to the restriction of C to

$$\{v \in S^{n-1} \mid v_1 = 0, \quad i \notin T \cup \{n\}\}.$$

K, $A(\cdot)$, $\mathcal{J}$, and N constitute a V-complex.

We first examine the set B. $\emptyset \in B$ since $A(\emptyset)$ contains only one 0-simplex. Suppose $\emptyset \neq x \in B$. Then $L(x) = T_x$ and $x \in \partial'A(T_x)$. Since $x \in \partial'A(T_x)$, $v_n = 0$ for all $v \in x$. And since x contains $|T_x|$ elements, $F(x) = T_x$. Thus $F(x) = L(x)$ and x is completely labelled in its face.

Next we examine G. Let $x \in G$. Then x is full, $L(x) \supset T_x$, and $L(x) \not\supset \mathcal{J}$. Then $L(x) = T_x \cup \{n\}$, and since x contains $|T_x| + 1$ elements, $F(x) = T_x \cup \{n\}$. Thus $L(x) = F(x)$.

Conversely, suppose $F(x) = L(x)$. Then if $n \in F(x)$, $x \in G$; if $n \notin F(x)$, $x \in B$. Therefore $B \cup G \setminus \emptyset$ is the set of simplices each of which is completely labelled in its face. Since B and G are disjoint and have the same parity by Lemma 13 of Part I, there are an odd number of simplices $x \in K$ such that x is completely labelled in its face. $\otimes$

An algorithm for computing an element of $G \cup B \setminus \emptyset$ consists of following the path of adjacent simplices where the endpoint is $\emptyset$. This is essentially Luthi's algorithm in [14].

As a corollary to the Generalized Sperner Lemma, we have:

Dual Sperner Lemma. (See Scarf [15], e.g.). Let $L(\cdot): K^0 \to \{1, \ldots, n\}$ be a labelling such that $v \in \partial S^{n-1}$, $v_i > 0$ implies $L(v) \neq i$, and suppose no simplex σ of C has a nonempty intersection with every face of S^{n-1}. Then there are an odd number of completely labelled simplices $x \in K$.

PROOF. By the Generalized Sperner Lemma, there are an odd number of simplices $x \in K$ such that x is completely labelled in its face. Let x be one such simplex. If $L(x) \neq \{1, \ldots, n\}$, then for all $i \notin L(x)$, $v_i = 0$ for all $v \in x$. And since $F(x) = L(x)$, each $v \in x$ meets the boundary of S^{n-1}, whence for each $i \in L(x)$, there is a $v \in x$ with $v_i = 0$, by design of $L(\cdot)$. Thus x meets every face of S^{n-1}, a contradiction. Therefore $L(x) = \{1, \ldots, n\}$. $\otimes$

An algorithm for finding the completely labelled simplex of this lemma is the same as that for the Generalized Sperner Lemma. Any element of $G \cup B \setminus \emptyset$ will be completely labelled.

The Generalized Sperner Lemma also implies the Sperner Lemma, where by "implies" we mean that one leads to simple straightforward proof of the other. We have

The Generalized Sperner Lemma implies the Sperner Lemma

PROOF: Our proof is by induction on n. For n = 1, the two lemmas are trivially identical. Suppose the implication is true for all $k < n$. Let $N = \{1, \ldots, n\}$, and for $\emptyset \neq T \subseteq N$, let $S^T = \{x \in \mathbb{R}^n \mid x \geq 0, e^t x = 1, x_i = 0 \text{ for } i \notin T\}$. Note that $S^{n-1} = S^N$, and note that if $L(\cdot)$ is a proper labelling of S^N, it is also a proper labelling of S^T, i.e. $v \in K^0 \cap S^T$ implies $L(v) \in T$. For $\emptyset \neq T \neq N$, we inductively have that there are an odd number of simplices in S^T whose set of labels is T. We have:

(number of simplices in S^{n-1} that are completely labelled)

$$+ \sum_{\substack{\emptyset \neq T \neq N \\ T \subseteq N}} (\# \text{ simplices in } S \text{ that have label set } T)$$

= an odd number

by the Generalized Sperner Lemma. Each term in the summation Σ is odd by induction, and there are $2^n - 2$ terms, an even number for $n \geq 1$. Thus the total number of simplices in the summation Σ is even, and so the number of simplices in S^{n-1} which are completely labelled is odd. $\otimes$

Each of the above three combinatorial lemmas bears an interesting relationship to the Brouwer Fixed Point Theorem [1], stated here for the simplex.

<u>Brouwer's Theorem on the Simplex</u>. Let $f: S^{n-1} \to S^{n-1}$ be a continuous function. Then there exists a point $v \in S^{n-1}$ such that $f(v) = v$. ⊗

That Scarf's Dual Sperner Lemma implies and is implied by the Brouwer Fixed Point Theorem has been shown in [15], for example. It has also been shown, see [11] for example, that Sperner's Lemma implies Brouwer's Theorem. We also have the following implications.

<u>The Generalized Sperner Lemma implies Brouwer's Theorem</u>

<u>PROOF</u>: Let $f: S^{n-1} \to S^{n-1}$ be continuous. Let C be a locally finite triangulation of S^{n-1}. Let $L(\cdot): K^0 \to \{1, \ldots, n\}$ be defined as

$$L(v) = \text{some } i \text{ such that } f_i(v) \geq v_i.$$

Let x be completely labelled in its face. Then for $i \in F(x) = L(x)$, $f_i(v) \geq v_i$ for some $v \in x$. And for $i \notin F(x)$, $v_i = 0$ for any $v \in x$ and so $f_i(v) \geq v_i$. Taking a sequence of finer triangulations and choosing a limiting subsequence of simplices x such that x is completely labelled in its face, we have a point v with $f_i(v) \geq v_i$ for all i, by the continuity of f. Since $e^t f(v) = e^t v = 1$, we must have $f(v) = v$. ⊗

<u>Brouwer's Theorem implies the Generalized Sperner Lemma</u>.

<u>PROOF</u>: Let $L(\cdot): K^0 \to \{1, \ldots, n\}$, and define $f(v) = e^{L(v)}$ for $v \in K^0$. Extend f in a piecewise linear (PWL) fashion on all of S^{n-1}. f is continuous and maps S^{n-1} into S^{n-1}. Thus there is a fixed point v of f. Let σ be the smallest real simplex in C containing v, and let x be the set of vertices of σ.

Then we must have $L(x) = F(x)$, whereby x is completely labelled in its face. ⊗

Finally we have:

Brouwer's Theorem implies the Sperner Lemma

PROOF: Let $L(\cdot): K^0 \to \{1, \ldots, n\}$ be a proper labelling. Define $f(v) = e^j$ where

$$j = \begin{cases} L(v) - 1 & \text{if} \quad L(v) \geq 2 \\ n & \text{if} \quad L(v) = 1 \end{cases}$$

for all $v \in K^0$, and extend f to all of S^{n-1} in a PWL fashion. f is continuous and maps S^{n-1} into S^{n-1}. Let v be a fixed point of f, let σ be the smallest simplex of C containing v, and let x be the set of vertices of σ. If $L(x) \neq \{1, \ldots, n\}$, there is an $i \in \{1, \ldots, n\}$ such that $i \notin L(x)$. Therefore f maps σ into the face of S^{n-1} opposite the vertex e^k, where

$$k = \begin{cases} i-1 & \text{if} \quad i \geq 2 \\ n & \text{if} \quad i = 1 \end{cases}.$$

But since $L(.)$ is proper, then k is not an element of $L(x)$. Continuing the argument, we obtain $L(x) = \emptyset$, a contradiction. Thus $L(x) = \{1, \ldots, n\}$, and x is completely labelled. ⊗

Figure 1 summarizes the relationships demonstrated herein or elsewhere, where the implication is in the direction of the arrow.

II. A Generalized Covering Lemma on the Simplex, and Extensions

In this section we prove a Generalized Covering Lemma on the simplex, "generalized" in that it generalizes the Knaster-Kuratowski-Mazurkiewicz Lemma [12]. We use this lemma to demonstrate the existence of stationary points (see Eaves [2], for example) and two other results on the simplex.

Generalized Covering Lemma. Let C^i, $i = 1, \ldots, n$ be closed sets in IR^n such that $\bigcup_{i=1}^{n} C^i \supset S^{n-1}$. Then there is at least one point x^* in S^{n-1} such that $\{i \mid x_i^* > 0\} \subset \{i \mid x^* \in C^i\}$.

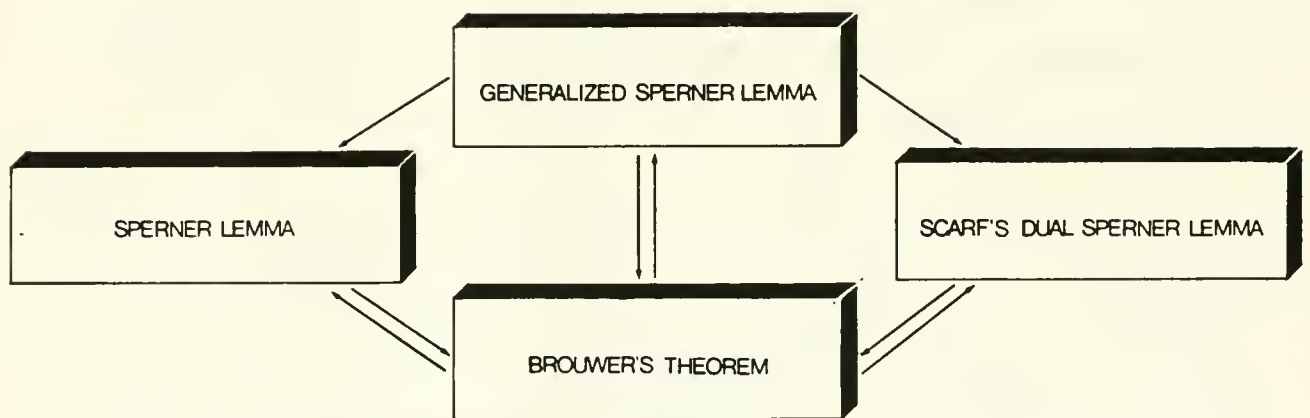Figure 1

PROOF: Let C be a locally finite triangulation of S^{n-1} with vertex set K^0, and let L(.) be a labelling of K^0, where for each $v \in K^0$,

$$L(v) \in \{i | v \in C^i\}.$$

By the Generalized Sperner Lemma, there is a simplex σ of C that is completely labelled in its face. Take a sequence of triangulations whose diameter goes to zero in the limit. Then there is a sequence of simplices σ, completely labelled in their faces, that have a limit point, say x^*. Since each C^i is a closed set, we have $\{i | x_i^* > 0\} \subset \{i | x^* \in C^i\}$, proving the theorem. $\otimes$

This theorem is illustrated in Figure 2. In Figure 2(b) a type of "degeneracy" occurs at x^*, showing that strict inclusion "$\subset$" of the theorem can indeed occur.

Our next lemma demonstrates the existence of stationary points (see Eaves [2], and Lüthi [14]). Let $S^n = \{(x,w) \in IR^{n+1} | e^t x + w = 1, x \geq 0, w \geq 0\}$, where it is understood that $x \in IR^n$. Let $D^n = \{x \in IR^n | e^t x \leq 1, x \geq 0\}$. Clearly D^n is the projection of S^n onto the x-coordinates. Let $f(\cdot):D^n \to IR^n$ be continuous. A point x in D^n is said to be a __stationary point__ of the pair (f, D^n) (see Eaves [3]) if and only if there exists $z \in IR$ and $y \in IR^n$ such that

    i) $y \geq 0, z \geq 0$

    ii) $f(x) = y - ze$

    iii) $x \cdot y = 0$
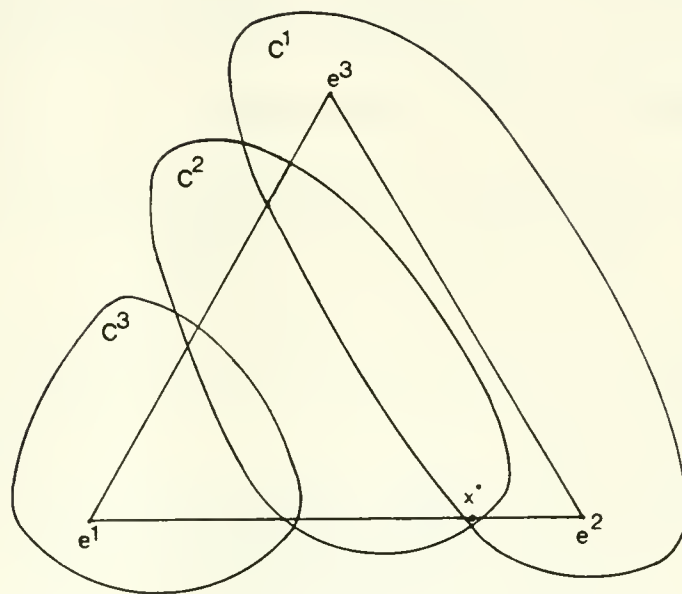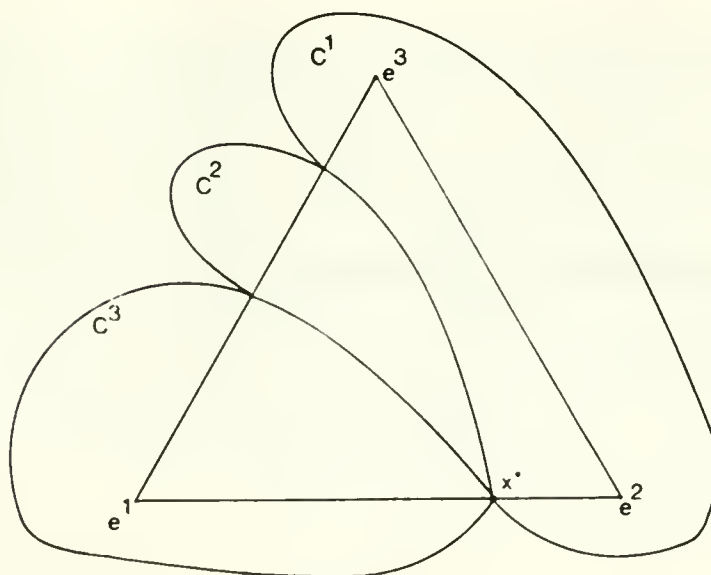
    iv) $z(1 - e^t x) = 0$

Figure 2(a)

Figure 2(b)

We have the following:

Lemma (Hartman and Stampacchia [7], and Karamardian [8a], [8b], [9]).
There exists a stationary point $x*$ of (f, D^n).

PROOF: Our proof is based on the Generalized Covering Lemma. For $i = 1, \ldots, n$, define

$$C^i = \{(x,w) \in S^n \mid f(x) \not> 0 \text{ and } f_i(x) \leq f_j(x) \text{ for any } j = 1, \ldots, n\}$$

Define $C^{n+1} = \{(x,w) \in S^n \mid f(x) \geq 0\}$. Note that each C^i ($i = 1, \ldots, n+1$) is closed, and that $\bigcup_{i=1}^{n+1} C^i \supset S^n$. Thus by the Generalized Covering Lemma, there exists $(x*, w*)$ in S^n such that

    i) $x_i^* > 0$ implies $(x*, w*) \in C^i$, $i = 1, \ldots, n$, and

    ii) $w* > 0$ implies $(x*, w*) \in C^{n+1}$.

We now show that $x*$ is a stationary point of (f, D^n). We have two cases:

Case I. $(x*, w*) \in C^{n+1}$. In this case, let $z = 0$, and let $y = f(x*)$.
$(x*, w*) \in C^{n+1}$ implies $f(x*) \geq 0$, and so $y \geq 0$. Also, $x_i^* > 0$ implies $f_i(x*) \leq 0$, but $f(x*) \geq 0$, therefore $f_i(x*) = 0$. Thus $x* \cdot y = x* \cdot f(x*) = 0$. Finally, $z(1 - e^T \cdot x*) = 0$ since $z = 0$. Therefore; $x*$ is a stationary point.

Case II. $(x*, w*) \notin C^{n+1}$. Let $z = -\min\{f_1(x*), \ldots, f_n(x*)\}$. Note that $z \geq 0$. Let $y = f(x*) + ze$. We have $-z \leq f_i(x*)$ for each i. Hence $y = f(x*) + ze \geq 0$. Furthermore, $x_i^* > 0$ implies $f_i(x*) \leq f_j(x*)$ for any j, and since $f(x*) \not\geq 0$, $f_i(x*) = -z$. Therefore $y_i = 0$, and so $x* \cdot y = 0$. Finally, since $(x*, w*) \notin C^{n+1}$, we must have $w* = 0$, so $e^T x* = 1$, which implies $z(1 - e^T x*) = 0$. Thus $x*$ is a stationary point of (f, D^n). $\otimes$

III. Combinatorial Lemmas on the Cube Related to Brouwer's Theorem

In this section, we present two new dual combinatorial lemmas on
the cube. We present and give constructive algorithmic proofs to Kuhn's
Cubical Lemma and Gale's Hex Theorem. We show the interrelationship between
these results and the Brouwer Fixed Point Theorem.

Let C be a locally finite triangulation of $C^n = \{x \in IR^n | 0 \le x \le e\}$
and let K be the pseudomanifold corresponding to C, with vertex set K^0. Our
first two results are the following:

Lemma 1   (Freund, [4]). Let $L(\cdot):K^0 \to \{1, \ldots, n, -1, \ldots, -n\}$ be a
labelling such that if $v \in K^0$,

$$v_i = 0 \text{ implies } L(v) \neq -i, \text{ and}$$
$$v_i = 1 \text{ implies } L(v) \neq i.$$

Then there exists a pair of vertices v', v'' in some simplex of K such
that $L(v') = -L(v'')$.

PROOF: Let $L(\cdot)$ be as described above. We proceed by defining a V-complex
and examining the sets B and G. Define $N = \{1, \ldots, n, -1, \ldots, -n\}$
and let $\mathcal{J} = \{S \subset N | i \in S \text{ implies } i > 0\}$. Define $A(\emptyset) = \{\emptyset, \{0\}\}$, and for
$\emptyset \neq T \in \mathcal{J}$, define $A(T)$ to be the pseudomanifold corresponding to the
restriction of C to

$$\{y \in C^n | y_i = 0 \text{ for } i \notin T\}.$$

K, $A(\cdot)$, $\mathcal{J}$, and N constitute a V-complex.

Examining B, observe that $\emptyset \in B$, since $A(\emptyset)$ contains only one 0-simplex. Suppose $\emptyset \neq x \in B$. Then $x \in \partial'A(T_x)$ and $L(x) = T_x$. Since $x \in \partial'A(T_x)$, there exists an $i \in T_x$ for which $v_i = 1$ for all $v \in x$. Then by the definition of the labelling rule $i \notin L(x)$, contradicting the fact that $L(x) = T_x$. Thus $B = \{\emptyset\}$.

Now let us examine G. Since $|B| = 1$, G must have an odd number of elements, by Lemma 13 of Part I. Let $x \in G$. Then $L(x) \supset T_x$ and $L(x) \notin \mathcal{J}$. Thus there is some $i \in N$ such that $-i \in L(x)$. Suppose $i \notin T_x$. Then $v_i = 0$ for all $v \in x$ and $-i \notin L(x)$ be definition of $L(\cdot)$. This is a contradiction. Thus $\{i, -i\} \in L(x)$ and there exists $\{v', v''\} \subset x$ such that $L(v') = -L(v'')$. $\otimes$

<u>Lemma 2</u>. Let $L(\cdot) : K \overset{0}{\to} \{1, \ldots, n, -1, \ldots -n\}$ be a labelling such that if $v \in K^0$ and $v \in \partial C^n$,

$$L(v) = i > 0 \text{ implies } v_i = 1$$
$$L(v) = -i < 0 \text{ implies } v_i = 0.$$

Then there exists a pair of vertices v', v'' in some simplex of K such that $L(v') = -L(v'')$.

<u>PROOF</u>: As in the proof of the Lemma 1, we proceed by defining a V-complex and examining the sets B and G. Let $N = \{1, \ldots, n, -1, \ldots, -n\}$ and let $\mathcal{J} = \{S \subset N \mid i \in S$ implies $i < 0\}$. Let $A(\emptyset) = \{\emptyset, \{0\}\}$ and for $\emptyset \neq T \in \mathcal{J}$, define $A(T)$ to be the pseudomanifold corresponding to the restriction of C to

$$\{y \in C^n \mid y_i = 0 \text{ for any } i > 0 \text{ and } -i \notin T\}.$$

K, $A(\cdot)$, $\mathcal{J}$, and N constitute a V-complex.

Examining B, we see that $\emptyset \in B$, since $A(\emptyset)$ contains only one

0-simplex. Suppose $\emptyset \neq x \in B$. Then $x \in \partial'A(T_x)$ and $L(x) = T_x$. Thus

there exists an $i > 0$ such that $-i \in T_x$ and $v_i = 1$ for all $v \in x$.

But then from the definition of $L(\cdot)$, we cannot have $L(v) = -i$ for any

$v \in x$, contradicting the fact that $-i \in T_x = L(x)$. Thus $x \notin B$ and

$B = \{\emptyset\}$.

Therefore G must have an odd number of elements, by Lemma 13 of

Part I. Suppose $x \in G$. Then $L(x) \supset T_x$ and $L(x) \notin \mathcal{J}$. Then there exists

$i > 0$ such that $i \in L(x)$. Thus $L(x) = T_x \cup \{i\}$. Suppose $-i \notin L(x)$.

Then $-i \notin T_x$ and so $v_i = 0$ for all $v \in x$. But then we cannot have

$i \in L(x)$ by the definition of $L(\cdot)$. Thus $-i \in L(x)$, $\{i, -i\} \subset L(x)$ and

there exists a pair of vertices $\{v', v''\}$ in x such that $L(v') = -L(v'')$. $\otimes$

An algorithm for finding a pair (v', v'') of vertices such that

$L(v') = -L(v'')$ in either of the two lemmas consists of following the path

of adjacent simplices from $\emptyset \in B$. The path must terminate with an element

of G, providing the desired pair.

It should be noted that Lemma 1 corresponds precisely to the existence of a

j-stopping simplex on the product space of n 1-simplices, as presented in [20].

We next give a proof of Gale's Hex Theorem [6]. Before doing so,

some terminology must be introduced. Let $L(\cdot):K^0 \to \{1, \ldots, n\}$ be a given

labelling of the vertices of K. Define $H_i = \{v \in K^0 | L(v) = i\}$. We say

H_i is <u>i-connected</u> if there is a sequence of vertices $v^0, \ldots, v^m$ such that

$v^j \in H_i$ for all $j = 0, \ldots, m$ and $\{v^{j-1}, v^j\}$ is a 1-simplex of K,

$j = 1, \ldots, m$, and $v_i^0 = 0$, $v_i^m = 1$. If H is i-connected, there is a

"path" of vertices from the boundary set $\{y \in C^n | y_i = 0\}$ to the

boundary set $\{y \in C^n | y_i = 1\}$ all of whose labels are i. We now prove:

<u>Hex Theorem [6]</u>. There exists an $i \in \{1, \ldots, n\}$ such that H_i is 1-connected.

<u>PROOF</u>: We proceed by defining a V-complex on K. Let $N = \{1, \ldots, n\}$ and let $\mathcal{J} = \{T | T \subset N\}$. Define $A(\emptyset) = \{\emptyset, \{0\}\}$. For $\emptyset \neq T \in \mathcal{J}$, define $A(T)$ to be the pseudomanifold corresponding to the restriction of C to

$$\{y \in C^n | y_i = 0 \text{ for } i \notin T\}.$$

K, $A(\cdot)$, $\mathcal{J}$, and N constitute a V-complex. Note that $G = \emptyset$ since $\mathcal{J}$ exhausts all subsets of N. We have $\emptyset \in B$, since $A(\emptyset)$ contains only one 0-simplex. Since K is finite there exists at least one other element of B, by Lemma 13 of Part I. Let $\emptyset = x^0, \ldots, x^m = x$ be the path of adjacent simplices from $\emptyset$ to $x \in B$. Since $x \in B$, there exists an $i \in T_x$ such that $v_i = 1$ for all $v \in x$. We show that H_i is 1-connected by exhibiting the desired sequence.

Let $T^j = T_{x^j}$, $j = 1, \ldots, m$. Since $i \in T_x = T^m$, there must be a number p such that $i \notin T^{p-1}$, and $i \in T^j$, $j = p, \ldots, m$. Since x^j is adjacent to x^{j+1} if and only if $L(x^j \cap x^{j+1}) = T^j \cup T^{j+1}$, the intersection of labels of any two consecutive x^j, x^{j+1} must contain i, for $j = p-1, \ldots, m$. Thus there is a sequence of vertices v^j in the x^j such that $\{v^j, v^{j+1}\}$ is a 1-simplex in K and $L(v^j) = i$, $j = p-1, \ldots, m$. Since T^{p-1} does not contain i, $v_i^{p-1} = 0$. Thus H_i is 1-connected. $\otimes$

This proof is different from that of Gale [6]. His is algorithmic like ours, but depends on augmenting C^n by adding another set of vertices to K and using a lexicographic labelling rule. He also restricts C to be the triangulation K_1 (see Todd [18]), whereas this proof is valid for any triangulation of C^n.

Our final lemma of this section is Kuhn's Cubical Lemma of [10].
Let $I = \{y \in IR^n | y_i = 0 \text{ or } 1, i = 1, \ldots, n\}$. We have:

Cubical Lemma (Kuhn [10]). Let $\ell(\cdot):K^0 \to I$ such that

$$v_i = 0 \text{ implies } \ell_i(v) = 0, \text{ and}$$
$$v_i = 1 \text{ implies } \ell_i(v) = 1.$$

Let $L(v)$ equal the number of leading zeroes of $\ell(v)$ for each $v \in K^0$.
Then there exists an odd number of simplices $x \in K$ such that $L(x) = \{0,1, \ldots, n\}$.

PROOF: We first construct a V-complex. Let $N = \{0, 1, \ldots, n\}$, and let
$\mathcal{J} = \{T \subset \{0, \ldots, n-1\} | 0 < i \in T \text{ implies } i-1 \in T\}$. $\mathcal{J}$ then is the collection

$$\emptyset, \{0\}, \{0,1\}, \{0,1,2\}, \ldots, \{0,1,\ldots,n-1\}.$$

We define $A(\emptyset) = \{\emptyset,\{e\}\}$, and for $\emptyset \neq T \in \mathcal{J}$, $T = \{0,\ldots, m\}$ for some
$m < n$. We then define $A(T)$ to be the pseudomanifold corresponding to the
restriction of C to

$$\{x \in C^n | x_i = 1 \text{ for } i > m+1\}.$$

It is simple to verify that $K, \mathcal{J}, A(\cdot)$, and N define a V-complex.

Let us now examine the set B. We know that $\emptyset \in B$. Suppose
$\emptyset \neq x \in B$, where $x = \{v^1, \ldots, v^m\}$ for some m. Then $T_x = \{0, \ldots,m-1\}$
and $L(x) = T_x$ and $x \in \partial'A(T_x)$. Since $x \in \partial'A(T_x)$, either $v_i^j = 0$ for
all $j = 1, \ldots, m$ and some $i \in \{1, \ldots, m\}$, or $v_i^j = 1$ for all $j = 1, \ldots, m$
and some $i \in \{1, \ldots, m\}$. Suppose the former is true. Then $i-1 \notin L(x)$,
a contradiction. If the latter is true, then $i \notin L(x)$, which is a contradiction

unless $i = m$. But then $x \in A(\{0, \ldots, m-2\})$, so that $T_x = \{0, \ldots, m-2\}$, a contradiction. Therefore $B = \{\emptyset\}$.

Next we examine the set G. Let $x \in G$. Then $T_x = \{0, \ldots, m\}$ for some $m < n$, and $L(x) \supset T_x$, $L(x) \notin \mathcal{J}$. Therefore either $L(x) = \{0, 1, \ldots, n\}$ or $m < n-1$ and there is an $s > m+1$ such that $L(x) = \{0, \ldots, m, s\}$. Suppose the latter is true. Let $x = \{v^0, \ldots, v^{m+1}\}$. Since $T_x = \{0, \ldots, m\}$, $v^j_{m+2} = 1$ for all $j \in \{0, \ldots, m+1\}$. But then $L(v^j) \le m+1$ for all j. This contradicts $s \ge m+1$. Therefore $L(x) = \{0, \ldots, n\}$.

Furthermore, if $L(x) = \{0, \ldots, n\}$, then clearly $x \in G$. Therefore G consists precisely of those x for which $L(x) = \{0, \ldots, n\}$. By Lemma 13 of Part I, G has an odd number of elements, which proves the lemma. $\otimes$

An algorithm for finding an element of G consists of following the path starting at $\emptyset \in B$, and terminating at its other endpoint, an element of G.

All of the above four results are closely related to Brouwer's Fixed Point Theorem on the cube, stated below.

Brouwer's Fixed Point Theorem on the Cube. Let $f: C^n \to C^n$ be continuous. Then there exists $v \in C^n$ such that $f(v) = v$. $\otimes$

In [6], Gale shows that the Hex Theorem implies and is implied by Brouwer's Theorem. In [10], Kuhn shows that his Cubical Lemma implies Brouwer's Theorem, and that Brouwer's Theorem implies a weaker version of his (strong) Cubical Lemma. We also have the following results:

Lemma 1 implies Brouwer's Theorem.

PROOF: Let $f: C^n \to C^n$ be continuous. Define $L(.): K^0 \to \{1, \ldots, n, -1, \ldots, -n\}$ by

$$L(v) = \begin{cases} i & \text{if } ||f(v) - v||_\infty = f_i(v) - v_i, \quad v_i \neq 1. \\ -i & \text{if } ||f(v) - v||_\infty = v_i - f_i(v), \quad v_i \neq 0. \end{cases}$$

If there is more than one choice for $L(v)$, let $L(v)$ be the smallest such index. $L(\cdot)$ satisfies the hypothesis of Lemma 1. Thus there exist v', v'' of K^0 in some simplex x of K for which $L(v') = -L(v'')$. Using the typical limiting argument as we take a sequence of finer triangulations, we obtain a point y in C^n such that $f(y) = y$. $\otimes$

A similar argument establishes that Lemma 2 implies Brouwer's Theorem, where we define $L(\cdot)$ by

$$L(v) = \begin{cases} i & \text{if } ||f(v) - v||_\infty = v_i - f_i(v) \\ -i & \text{if } ||f(v) - v||_\infty = f_i(v) - v_i \end{cases}$$

If there is more than one choice for $L(v)$, choose the smallest such index, ensuring that if $v \in \partial C^n$, the hypothesis of Lemma 2 is satisfied.

Finally, we show the equivalance of Lemma 1 and the Hex Theorem.

The Hex Theorem implies Lemma 1. Let $L(\cdot): K^0 \to \{1, \ldots, n, -1, \ldots, -n\}$ be as described in Lemma 1. Define $L'(v) = |L(v)|$ and let $H_i = \{v \in K^0 | L'(v) = i\}$, $i = 1, \ldots, n$. By the Hex Theorem, at least one H_i is i-connected. Thus there is a sequence $v^0, \ldots, v^m$ of vertices such that $L'(v^j) = i$, $j = 0, \ldots, m$, and $\{v_{j-1}, v_j\}$ is a 1-simplex, $j = 1, \ldots, m$. Also $v_i^0 = 0$ and $v_i^m = 1$. By the definition of $L(\cdot)$, we have neither $L(v^0) = -i$, nor $L(v^m) = i$, so that $L(v^0) = i$ and $L(v^m) = -i$. Thus there must be some $j \in \{0, \ldots, m-1\}$ with $L(v^j) = -i$, $L(v^{j+1}) = i$. $\otimes$

Lemma 1 implies the Hex Theorem. Let $L(\cdot):K^0 \to \{1, \ldots, n\}$ be given and define the sets $H_i = \{v \in K^0 | L(v) = i\}$. An element v of H_i is defined to be __base-connected__ if there is a sequence of 1-simplices $\{v^0, v^1\}$, $\ldots$, $\{v^{m-1}, v^m\}$ in K for which $v^j \in H_i$, $j = 0, \ldots, m$, $v_i^0 = 0$, and $v = v^m$. Define $L'(v):K^0 \to \{1, \ldots, n, -1, \ldots, -n\}$ as follows:

$$L'(v) = \begin{cases} L(v) & \text{if } v \text{ is base-connected.} \\ -L(v) & \text{if } v \text{ is not base-connected.} \end{cases}$$

Note that if $v_i = 0$, we cannot have $L'(v) = -i$.

Suppose the Hex Theorem is false. Then if $v_i = 1$, we cannot have $L'(v) = i$. Thus $L'(\cdot)$ satisfies the hypothesis of Lemma 1. Therefore there exists a 1-simplex $\{v', v''\} \in K$ such that $L'(v') = -L'(v'')$. Without loss of generality, assume $L'(v') = i > 0$. Since $\{v', v''\}$ is a 1-simplex and $\{v', v''\} \subset H_i$, v'' is base-connected because v' is base-connected. This is a contradiction, since $L'(v'') = -i < 0$. Therefore the Hex Theorem must be true. $\otimes$

Our two final results of this section are:

Brouwer's Theorem implies Lemma 1.

__PROOF:__ Let $L(\cdot)$ be a labelling of K^0 as in Lemma 1. Define $f(v)$, for $v \in K^0$ as

$$f(v) = \begin{cases} (v_1, \ldots, v_{i-1}, 0, v_{i+1}, \ldots, v_m) & \text{if } L(v) = -i < 0 \\ (v_1, \ldots, v_{i-1}, 1, v_{i+1}, \ldots, v_m) & \text{if } L(v) = i > 0 \end{cases}$$

and extend $f(\cdot)$ in a PWL fashion over all of C^n. By the definition of $L(\cdot)$ in Lemma 1, we cannot have $f(v) = v$ for $v \in K^0$. Since f is continuous

and maps C^n into itself, there exists a fixed point y of f. Let σ be a simplex of C containing y, and let x be the abstract simplex corresponding to σ. $L(x) \supset \{-i, i\}$ for some $i \in \{1, \ldots, n\}$, otherwise y cannot be a fixed point of f. $\otimes$

Similarly, we have

Brouwer's Theorem implies Lemma 2

PROOF: The proof is analogous to the one above, except we define

$$
f(v) = \begin{cases} (v_1, \ldots, v_{i-1}, 0, v_{i+1}, \ldots, v_n) & \text{if } L(v) = i > 0. \\ (v_1, \ldots, v_{i-1}, 1, v_{i+1}, \ldots, v_n) & \text{if } L(v) = -i < 0. \end{cases}
$$

$$\text{Q.E.D. } \otimes$$

The interrelationships shown in this section are summarized in Figure 3.

IV. Tucker's Lemma and Antipodal Point Theorems

In this section we present a proof of Tucker's Lemma [19] on the cube and show its relationship to two antipodal point theorems.

Let p be a positive integer and let $C_p^n = \{x \in IR^n \mid -pe \le x \le pe\}$. Let C be any locally finite centrally symmetric triangulation of C_p^n that refines the octahedral subdivision, for example J_1, see Todd [18]. Let K^0 be the set of vertices of C and K the pseudomanifold corresponding to C. For $x \in K$, we denote by $-x$ the simplex $\{-v \mid v \in x\}$. By symmetry, $-x \in K$. A function $L(.)$ whose domain is K^0 is called odd if $L(v) = -L(-v)$ for all $v \in K^0$. Our first result is:
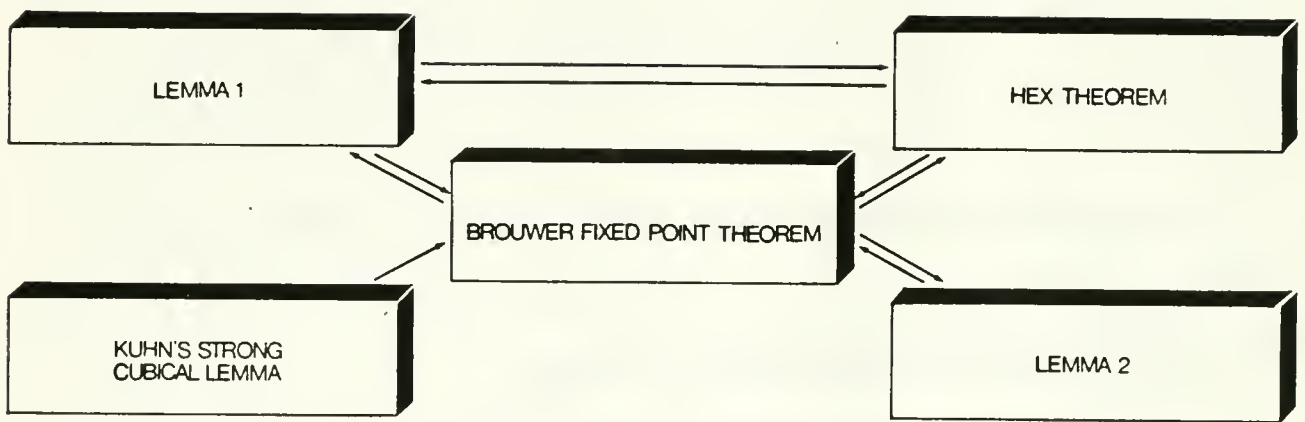
Figure 3

<u>Tucker's Combinational Lemma</u>. (see Tucker [19]). Let $L(\cdot):K^0 \to \{1, \ldots, n, -1, \ldots, -n\}$ such that $L(.)$ is odd on ∂K. Then there exists a 1-simplex $\{v', v''\}$ of K such that $L(v) = -L(v'')$.

<u>PROOF</u>: We first construct a V-complex. Let $N = \{1, \ldots, n, -1, \ldots, -n\}$, and let $\mathcal{J} = \{T \subset N | i \in T \text{ implies } -i \notin T\}$. Let $A(\emptyset) = \{\emptyset, \{0\}\}$, and for $\emptyset \neq T \in \mathcal{J}$, let $A(T)$ be the pseudomanifold corresponding to the restriction of C to the region

$$\{x \in \mathbb{R}^n | ix_{|i|} \geq 0 \text{ for } i \in T, \text{ and } x_i = 0 \text{ if neither } i \text{ nor } -i \in T\}.$$

It is simple to verify that $K, \mathcal{J}$, $A(\cdot)$, and N define a V-complex.

Let us now examine the set B. $\emptyset \in B$, so B has at least one element. Suppose $\emptyset \neq x \in B$. Then $x \in \partial' A(T_x)$. For any $\emptyset \neq T \in \mathcal{J}$, $\partial' A(T) \subset \partial K$.

Therefore, $x \in \partial K$. Also $L(x) = T_x$. Furthermore, $-x \in \partial K$, and $L(-x) = \{-i | i \in T_x\}$, and in fact $T_{-x} = \{-i | i \in T_x\}$. Therefore, $-x \in B$. Thus we see that except for $\emptyset$, B consists of pairs of the form $x, -x$. Therefore B has an odd number of elements, and so must G, by Lemma 13 of Part I.

Thus there is an element $x \in G$. Thus there are two vertices of x, say v' and v'', such that $L(v') = -L(v'')$. And since v' and v'' are elements of x, $\{v', v''\}$ is a 1-simplex of K. $\bigotimes$

An algorithm for finding a pair v', v'' consists of following the path that originates with $\emptyset$. If its endpoint is an element of G, stop. If it is an element x of B, reinitiate the path at $-x$. Continuing in this fashion, an element of G will be found. For a complete description

of the pivot rules for this algorithm, see Freund and Todd [5]. We cannot assert that there are an odd number of 1-simplices {v', v"} with $L(v') = -L(v")$. This is because not all such pairs are subsets of elements of G.

That Tucker's Lemma implies both the Borsuk-Ulam Theorem (see [13]) and the Lusternik-Schnirelman Theorem (see [13]), and that the Borsuk-Ulam Theorem is implied by the Lusternik-Schnirelman Theorem, have been shown elsewhere, see Lefschetz [13] or Freund and Todd [5] for more familiar terminology. Here we show that the Borsuk-Ulam Theorem implies the Tucker Combinatorial Lemma. The former theorem can be stated:

<u>Borsuk-Ulam Theorem</u>: Let $B^{n-1} = \{x \in \mathbb{R}^n | \ ||x||_2 = 1\}$, and let $f(\cdot):B^{n-1} \to \mathbb{R}^{n-1}$ be continuous. Then there exists $x* \in B^{n-1}$ such that $f(x*) = f(-x*)$.

We have :

The Borsuk-Ulam Theorem implies Tucker's Combinatorial Lemma

<u>PROOF</u>: Let $C_p^n = \{x \in \mathbb{R}^n | -pe \le x \le pe\}$ and let C be a locally finite centrally symmetric triangulation of C_p^n. Let K be the pseudomanifold corresponding to C, with vertex set K^0. Let $L(\cdot):K^0 \to \{1, \ldots, n, -1, \ldots, -n\}$ be a labelling function which is odd on the boundary of C_p^n. For each $v \in K^0$, define $f(v) = \text{sign}(L(v)).e^{|L(v)|}$, where $|.|$ denotes absolute value, and extend $f(.)$ in a piece-wise linear manner over all of C_p^n. Note that $f(.)$ is continuous, and since C is symmetric, $f(.)$ is odd on the boundary of C_p^n.

Let $B^n = \{x \in IR^{n+1} \mid \|x\|_2 = 1\}$, let $B^{n+} = \{x \in B^n \mid x_{n+1} \geq 0\}$, and let $B^{n-} = \{x \in B^n \mid x_{n+1} \leq 0\}$. Let $g: B^{n+} \to C_p^n$ be the following map:

$$g(x) = \begin{cases} \dfrac{p(x_1, \ldots, x_n) \|(x_1, \ldots, x_n)\|_2}{\|(x_1, \ldots, x_n)\|_\infty}, & (x_1, \ldots, x_n) \neq 0. \\[2ex] 0, & (x_1, \ldots, x_n) = 0. \end{cases}$$

Note that $g(\cdot)$ is bicontinuous and onto. For $x \in B^n$, let

$$h(x) = \begin{cases} f(g(x)), & x \in B^{n+} \\[2ex] -f(g(-x)), & x \in B^{n-} \end{cases}$$

$h(\cdot)$ is an odd continuous function from B^n into IR^n. By the Borsuk-Ulam Theorem, there exists $x*$ such that $h(x*) = h(-x*)$. Without loss of generality, we may assume $x* \in B^{n+}$. Thus $h(x*) = 0$, whereby $f(g(x*)) = 0$. Setting $\bar{x} = g(x*)$, we see there exists $\bar{x} \in C_p^n$ such that $f(\bar{x}) = 0$. $f(\bar{x}) = \Sigma \lambda_i$ sign $(L(v^i)).e^{|L(v^i)|}$ for appropriate v^i and $\lambda_i \geq 0$. Thus there must be a pair of vertices v^{i_1}, v^{i_2} such that $L(v^{i_1}) = -L(v^{i_2})$, proving Tucker's Lemma. $\otimes$

Final Comment

The algorithms presented herein for proving the Sperner Lemma, the Dual Sperner Lemma, and Lemmas 1 and 2, all follow a path that is initiated at a "corner" of the simplex or cube, i.e., the unit vector e^1 or the origin, respectively. As is shown in [4], it is possible to define a V-complex where the initiation point (corresponding to $A(\emptyset)$) is an interior grid point of the simplex or cube for the above lemmas if the triangulation is of a standard form. When the V-complex is defined this way, the resulting algorithms correspond completely to those of van der Laan and Talman [20].

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